RECOGNISING ACHIEVEMENT

ADVANCED GCE **4763/01**

MATHEMATICS (MEI)

Mechanics 3

THURSDAY 17 JANUARY 2008 Afternoon

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

[Turn over

1 **(a)** **(i)** Write down the dimensions of force and the dimensions of density. [2]

When a wire, with natural length l_0 and cross-sectional area A, is stretched to a length l, the tension F in the wire is given by

$$F = \frac{EA(l - l_0)}{l_0}$$

where E is Young's modulus for the material from which the wire is made.

(ii) Find the dimensions of Young's modulus E. [3]

A uniform sphere of radius r is made from material with density ρ and Young's modulus E. When the sphere is struck, it vibrates with periodic time t given by

$$t = kr^{\alpha}\rho^{\beta}E^{\gamma}$$

where k is a dimensionless constant.

(iii) Use dimensional analysis to find α, β and γ. [5]

(b) Fig. 1 shows a fixed point A that is 1.5 m vertically above a point B on a rough horizontal surface. A particle P of mass 5 kg is at rest on the surface at a distance 0.8 m from B, and is connected to A by a light elastic string with natural length 1.5 m.

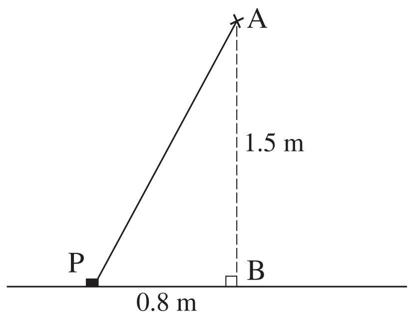

Fig. 1

The coefficient of friction between P and the surface is 0.4, and P is on the point of sliding. Find the stiffness of the string. [8]

2 **(a)** A small ball of mass 0.01 kg is moving in a vertical circle of radius 0.55 m on the smooth inside surface of a fixed sphere also of radius 0.55 m. When the ball is at the highest point of the circle, the normal reaction between the surface and the ball is 0.1 N. Modelling the ball as a particle and neglecting air resistance, find

(i) the speed of the ball when it is at the highest point of the circle, [3]

(ii) the normal reaction between the surface and the ball when the vertical height of the ball above the lowest point of the circle is 0.15 m. [5]

(b) A small object Q of mass 0.8 kg moves in a circular path, with centre O and radius r metres, on a smooth horizontal surface. A light elastic string, with natural length 2 m and modulus of elasticity 160 N, has one end attached to Q and the other end attached to O. The object Q has a constant angular speed of ω rad s^{-1}.

(i) Show that $\omega^2 = \frac{100(r-2)}{r}$ and deduce that $\omega < 10$. [4]

(ii) Find expressions, in terms of r only, for the elastic energy stored in the string, and for the kinetic energy of Q. Show that the kinetic energy of Q is greater than the elastic energy stored in the string. [4]

(iii) Given that the angular speed of Q is 6 rad s^{-1}, find the tension in the string. [3]

3 A particle is oscillating in a vertical line. At time t seconds, its displacement above the centre of the oscillations is x metres, where $x = A\sin\omega t + B\cos\omega t$ (and A, B and ω are constants).

(i) Show that $\frac{d^2x}{dt^2} = -\omega^2 x$. [3]

When $t = 0$, the particle is 2 m *above* the centre of the oscillations, the velocity is 1.44 m s^{-1} *downwards*, and the acceleration is 0.18 m s^{-2} *downwards*.

(ii) Find A, B and ω. [6]

(iii) Show that the period of oscillation is 20.9 s (correct to 3 significant figures), and find the amplitude. [3]

(iv) Find the total distance travelled by the particle between $t = 12$ and $t = 24$. [5]

[Question 4 is printed overleaf.]

4 Fig. 4.1 shows the region R bounded by the curve $y = x^{-\frac{1}{3}}$ for $1 \leqslant x \leqslant 8$, the x-axis, and the lines $x = 1$ and $x = 8$.

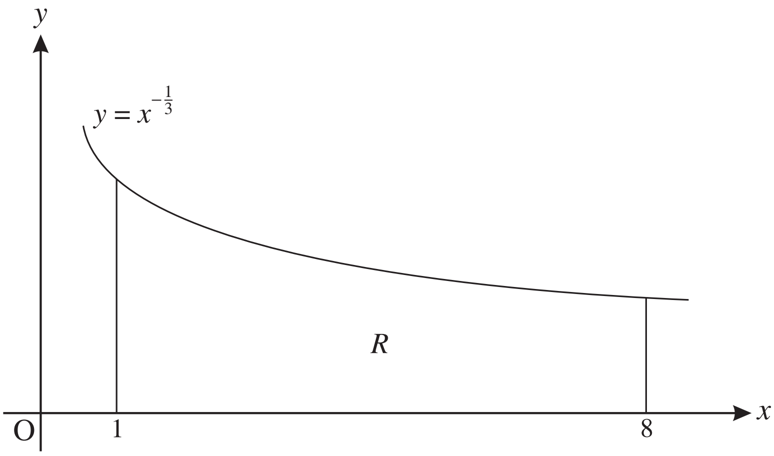

Fig. 4.1

(i) Find the x-coordinate of the centre of mass of a uniform solid of revolution obtained by rotating R through 2π radians about the x-axis. [6]

(ii) Find the coordinates of the centre of mass of a uniform lamina in the shape of the region R. [8]

(iii) Using your answer to part **(ii)**, or otherwise, find the coordinates of the centre of mass of a uniform lamina in the shape of the region (shown shaded in Fig. 4.2) bounded by the curve $y = x^{-\frac{1}{3}}$ for $1 \leqslant x \leqslant 8$, the line $y = \frac{1}{2}$ and the line $x = 1$. [4]

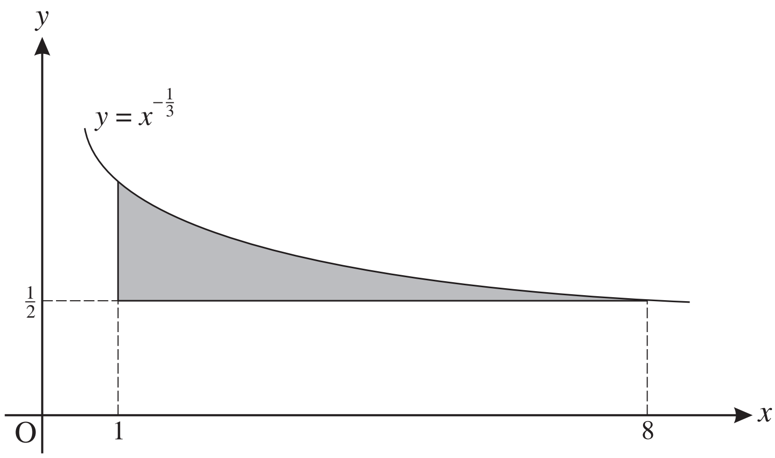

Fig. 4.2

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1(a)(i)	[Force] $= \mathrm{M\,L\,T^{-2}}$ [Density] $= \mathrm{M\,L^{-3}}$	B1 B1 **2**	
(ii)	$[E] = \dfrac{[F][l_0]}{[A][l - l_0]} = \dfrac{(\mathrm{M\,L\,T^{-2}})(\mathrm{L})}{(\mathrm{L^2})(\mathrm{L})}$ $= \mathrm{M\,L^{-1}\,T^{-2}}$	B1 M1 A1 **3**	for $[A] = \mathrm{L}^2$ Obtaining the dimensions of *E*
(iii)	$\mathrm{T} = \mathrm{L}^{\alpha}(\mathrm{M\,L^{-3}})^{\beta}(\mathrm{M\,L^{-1}\,T^{-2}})^{\gamma}$ $-2\gamma = 1, \quad \beta + \gamma = 0$ $\gamma = -\frac{1}{2}$ $\beta = \frac{1}{2}$ $\alpha - 3\beta - \gamma = 0$ $\alpha = 1$	B1 cao F1 M1 A1 A1 **5**	Obtaining equation involving α, β, γ
(b)	$AP = 1.7$ m $F = T\cos\theta$ $R + T\sin\theta = 5 \times 9.8$ $T\cos\theta = 0.4(49 - T\sin\theta)$ $\frac{8}{17}T = 0.4(49 - \frac{15}{17}T)$ $T = 23.8$ $T = k(1.7 - 1.5)$ Stiffness is $119\ \mathrm{N\,m^{-1}}$	B1 M1 M1 M1 A1 A1 M1 A1 **8**	Resolving in any direction Resolving in another direction *(M1 for resolving requires no force omitted, with attempt to resolve all appropriate forces)* Using $F = 0.4R$ to obtain an equation involving just one force (or *k*) Correct equation *Allow* $T\cos 61.9$ *etc* or $R = 28$ or $F = 11.2$ *May be implied* *Allow M1 for* $T = \dfrac{\lambda}{1.5} \times 0.2$ If $R = 49$ is assumed, max marks are B1M1M0M0A0A0M1A0

2(a)(i)	$0.1 + 0.01\times 9.8 = 0.01\times\dfrac{u^2}{0.55}$ Speed is $3.3\ \mathrm{m\,s^{-1}}$	M1 A1 A1 **3**	Using acceleration $u^2/0.55$
(ii)	$\tfrac{1}{2}m(v^2-u^2) = mg(2\times 0.55 - 0.15)$ $\tfrac{1}{2}(v^2-3.3^2) = 9.8\times 0.95$ $v^2 = 29.51$ $R - mg\cos\theta = m\dfrac{v^2}{a}$ $R - 0.01\times 9.8\times\dfrac{0.4}{0.55} = 0.01\times\dfrac{29.51}{0.55}$ Normal reaction is 0.608 N	M1 A1 M1 A1 A1 **5**	Using conservation of energy *(ft is* $v^2 = u^2 + 18.62$ *)* Forces and acceleration towards centre *(ft is* $\dfrac{u^2+22.54}{55}$ *)*
(b)(i)	$T = 0.8r\omega^2$ $T = \dfrac{160}{2}(r-2)$ $\omega^2 = \dfrac{80(r-2)}{0.8r} = \dfrac{100(r-2)}{r}$ $\omega^2 = 100 - \dfrac{200}{r} < 100,\ \text{ so }\ \omega < 10$	B1 B1 E1 E1 **4**	
(ii)	$\mathrm{EE} = \dfrac{1}{2}\times\dfrac{160}{2}\times(r-2)^2 = 40(r-2)^2$ $\mathrm{KE} = \tfrac{1}{2}m(r\omega)^2$ $= \tfrac{1}{2}\times 0.8\times r^2\times\dfrac{100(r-2)}{r}$ $= 40r(r-2)$ Since $r > r-2$, $40r(r-2) > 40(r-2)^2$ i.e. KE > EE	B1 M1 A1 E1 **4**	Use of $\tfrac{1}{2}mv^2$ with $v = r\omega$ From fully correct working only
(iii)	When $\omega = 6$, $36 = \dfrac{100(r-2)}{r}$ $r = 3.125$ $T = 80(r-2) = 80(3.125-2)$ Tension is 90 N	M1 M1 A1 cao **3**	Obtaining *r*

3 (i)	$\frac{dx}{dt} = A\omega\cos\omega t - B\omega\sin\omega t$ $\frac{d^2x}{dt^2} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$ $= -\omega^2(A\sin\omega t + B\cos\omega t) = -\omega^2 x$	B1 B1 ft E1 **3**	*Must follow from their* $\dot{x}$ Fully correct completion *SR* For $\dot{x} = -A\omega\cos\omega t + B\omega\sin\omega t$ $\ddot{x} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$ award B0B1E0
(ii)	$B = 2$ $A\omega = -1.44$ $-B\omega^2 = -0.18$ *or* $-0.18 = -\omega^2(2)$ $\omega = 0.3,\quad A = -4.8$	B1 M1 A1 cao M1 A1 cao A1 cao **6**	Using $\frac{dx}{dt} = -1.44$ when $t = 0$ $\frac{d^2x}{dt^2} = -0.18$ when $t = 0$ (or $x = 2$)
(iii)	Period is $\frac{2\pi}{\omega} = \frac{2\pi}{0.3} = 20.94 = 20.9$ s (3 sf) Amplitude is $\sqrt{A^2 + B^2} = \sqrt{4.8^2 + 2^2}$ $= 5.2$ m	E1 M1 A1 **3**	or $1.44^2 = 0.3^2(a^2 - 2^2)$
(iv)	$x = -4.8\sin 0.3t + 2\cos 0.3t$ $v = -1.44\cos 0.3t - 0.6\sin 0.3t$ When $t = 12,\ x = 0.3306$ ($v = 1.56$) When $t = 24,\ x = -2.5929$ ($v = -1.35$) Distance travelled is $(5.2 - 0.3306) + 5.2 + 2.5929$ $= 12.7$ m	M1 A1 M1 M1 A1 **5**	Finding *x* when $t = 12$ and $t = 24$ Both displacements correct Considering change of direction Correct method for distance ft from their A, B, ω and amplitude: *Third M1 requires the method to be comparable to the correct one* *A1A1 both require* $\omega \approx 0.3,\ A \neq 0,\ B \neq 0$ *Note* ft from $A = +4.8$ is $x_{12} = -3.92$ $(v < 0)$ $x_{24} = 5.03$ $(v > 0)$ Distance is $(5.2 - 3.92) + 5.2 + 5.03$ $= 11.5$

4 (i)	$V=\int_1^8 \pi(x^{-\frac{1}{3}})^2\,dx$	M1	*π may be omitted throughout*
	$=\pi\left[\,3x^{\frac{1}{3}}\,\right]_1^8=3\pi$	A1	
	$V\bar{x}=\int_1^8 \pi x(x^{-\frac{1}{3}})^2\,dx$	M1	
	$=\pi\left[\,\frac{3}{4}x^{\frac{4}{3}}\,\right]_1^8=\frac{45}{4}\pi$	A1	
	$\bar{x}=\dfrac{\frac{45}{4}\pi}{3\pi}$	M1	*Dependent on previous M1M1*
	$=\frac{15}{4}=3.75$	A1 **6**	
(ii)	$A=\int_1^8 x^{-\frac{1}{3}}\,dx$	M1	
	$=\left[\,\frac{3}{2}x^{\frac{2}{3}}\,\right]_1^8=\frac{9}{2}=4.5$	A1	
	$A\bar{x}=\int_1^8 x(x^{-\frac{1}{3}})\,dx$	M1	
	$=\left[\,\frac{3}{5}x^{\frac{5}{3}}\,\right]_1^8=\frac{93}{5}=18.6$	A1	
	$\bar{x}=\frac{18.6}{4.5}=\frac{62}{15}\quad(\approx 4.13)$	A1	
	$A\bar{y}=\int_1^8 \frac{1}{2}(x^{-\frac{1}{3}})^2\,dx$	M1	If $\frac{1}{2}$ omitted, award M1A0A0
	$=\left[\,\frac{3}{2}x^{\frac{1}{3}}\,\right]_1^8=\frac{3}{2}=1.5$	A1	
	$\bar{y}=\frac{1.5}{4.5}=\frac{1}{3}$	A1 **8**	

(iii)	$(1)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}+(3.5)\begin{pmatrix}4.5\\ 0.25\end{pmatrix}=(4.5)\begin{pmatrix}62/15\\ 1/3\end{pmatrix}=\begin{pmatrix}18.6\\ 1.5\end{pmatrix}$ $\bar{x}=2.85$ $\bar{y}=0.625$	M1 M1 A1 A1	 **4**	Attempt formula for CM of composite body (one coordinate sufficient) Formulae for both coordinates; signs must now be correct, but areas (1 and 3.5) may be wrong. ft only if $1<\bar{x}<8$ ft only if $0.5<\bar{y}<1$ *Other methods:* M1A1 for $\bar{x}$ M1A1 for $\bar{y}$ *(In each case, M1 requires a complete and correct method leading to a numerical value)*

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General Comments

Most candidates performed well on this paper, and were able to demonstrate a sound working knowledge of the topics being examined. Very few seemed to have any difficulty completing the paper in the time allowed. Nearly half the candidates scored 60 marks or more (out of 72) and very few scored fewer than half marks.

Comments on Individual Questions

1) This question (on dimensional analysis and elasticity) was very well answered. About 40% of candidates scored full marks, and the average mark was about 15 (out of 18).

(a) In part (i), the dimensions of force were almost invariably given correctly, but quite a few candidates gave the wrong dimensions for density.
In part (ii), the dimensions of Young's modulus were usually found correctly, although some thought that $(l - l_0)$ was dimensionless.
In part (iii), the method for finding the powers was well understood and frequently carried out accurately. Previous errors often trivialised the problem, so that follow-through marks could not always be awarded.

(b) Those who wrote down the correct equations from resolving horizontally and vertically were usually able to manipulate these, together with the condition for limiting friction and Hooke's law, to obtain the stiffness of the string correctly. However, a very common error was to assume that the normal reaction was equal to the weight of the particle P.

2) Only about 10% of candidates scored full marks on this question (on circular motion), and the average mark was about 14 (out of 19).

(a) In part (i), the speed of the ball at its highest point was very often found correctly, although the weight often appeared in the equation of motion with the wrong sign, and was sometimes omitted altogether.
Part (ii) caused many difficulties for the candidates. Although the correct methods were seen fairly often, sign errors and other careless slips frequently spoilt the work. Some candidates did not consider energy at all, assuming either that the speed was constant or that the vertical forces were in equilibrium.

(b) Most candidates had no difficulty obtaining the result in part (i), although the deduction that $\omega < 10$ was often missing or incomplete; it was not sufficient just to show that $\omega \to 10$ as $r \to \infty$.
In part (ii), the expression for elastic energy was usually found correctly, although some forgot the factor ½ and some forgot to square the extension. It was quite common for the kinetic energy to be taken as $\frac{1}{2} m\omega^2$ instead of $\frac{1}{2} m(r\omega)^2$. Even when both expressions were correct, many candidates could not show that the kinetic energy was greater than the elastic energy.
In part (iii), most candidates were able to find the tension correctly.

3) About 20% of the candidates scored full marks on this question (on simple harmonic motion), and the average mark was about 12 (out of 17).
Most candidates obtained the result in part (i) correctly.
In part (ii), the values of *A, B* and ω were usually found correctly. The most common error was to obtain *A* = +4.8 instead of *A* = -4.8, resulting from an incorrect sign for the initial velocity.
In part (iii), almost all candidates obtained the given period. The majority also found the amplitude correctly. The most common methods were to use $\sqrt{A^2 + B^2}$ or apply $v^2 = \omega^2(a^2 - x^2)$ to the initial values; a third method, finding the maximum value of x by differentiation, was occasionally used successfully. Some candidates did not seem to know how to find the amplitude, or assumed that it was A.
In part (iv), most candidates evaluated *x* when *t* = 12 and when *t* = 24 (although some were working in degrees instead of radians). However, very many simply subtracted these values to calculate the distance travelled. Those who did consider the change in direction of the particle's motion quite often obtained the correct answer. Some candidates rather strangely integrated the expression for x between *t* = 12 and *t* = 24.

4) This question (on centres of mass) was answered very well, with about 40% of candidates scoring full marks, and an average mark of about 15 (out of 18).
The techniques for finding the centre of mass of a solid of revolution in part (i), and of a lamina in part (ii), were very well understood and usually applied accurately (although there were some errors in integration and evaluation).
In part (iii), most candidates knew how to find the centre of mass of the shaded region obtained by removing a rectangle from the lamina R. However, a very common error here was to take the area of the rectangle to be 4 instead of 3.5. This led to $\bar{y} = 1$, which is very obviously wrong, but the candidates did not appear to be at all concerned.